Energetics of Actively Powered Locomotion Using the Simplest Walking Model

Arthur D. Kuo
Dept. of Mechanical Engineering and Applied Mechanics,
University of Michigan,
Ann Arbor, MI 48109-2125
email: artkuo@umich.edu

We modified an irreducibly simple model of passive dynamic walking to walk on level ground, and used it to study the energetics of walking and the preferred relationship between speed and step length in humans. Powered walking was explored using an impulse applied at toe-off immediately before heel strike, and a torque applied on the stance leg. Although both methods can supply energy through mechanical work on the center of mass, the toe-off impulse is four times less costly because it decreases the collision loss at heel strike. We also studied the use of a hip torque on the swing leg that tunes its frequency but adds no propulsive energy to gait. This spring-like actuation can further reduce the collision loss at heel strike, improving walking energetics. An idealized model yields a set of simple power laws relating the toe-off impulses and effective spring constant to the speed and step length of the corresponding gait. Simulations incorporating nonlinear equations of motion and more realistic inertial parameters show that these power laws apply to more complex models as well. [DOI: 10.1115/1.1427703]

1 Introduction

Human walking is a mechanically complex task that is powered by the activity of numerous muscles. This complexity makes it difficult to discern what principles govern the cost of transport. Simple models of walking have shown, however, that there are general principles that hold, such as the fact that the motion of the swing leg can be largely passive in nature, requiring little muscle activity (e.g., Mochon and McMahon [1]). Alexander [2] argues that especially simple models can lead to powerful insights regarding the dynamics of walking and its associated energetics.

A particularly parsimonious approach to the mechanics of walking is to study the passive dynamic properties of both the stance and swing legs together. McGeer [3] showed that a mechanism with two legs can be constructed so as to descend a gentle slope with no actuation and no active control. The mechanical energy consumption of such a mechanism is due primarily to the energy lost as the swing leg impacts with ground at heel strike. This loss may be compensated for by gravity, or in the case of level ground walking, by two general methods of actuation (as explained in [3]): One is to apply an impulsive push along the trailing leg, preferably immediately before heel strike for minimal energetic cost. The other method is to apply a hip torque against the stance leg, using the torso as a base.

A comparison of actuation methods for (quasi-) passive dynamic walking may yield insight regarding the mechanics of human walking. The passive nature of leg swing in McGeer's [3] model makes it clear where energy sinks occur and simplifies the analysis of energy input. A further advantage is gained by studying an irreducibly simple two-link model proposed by Garcia et al. [4], in which the inertia of the legs approaches zero relative to that of a point mass pelvis. This model exhibits the same passive dynamic characteristics found in McGeer's [3] more complex models, but is more tractable from an analytical standpoint [5].

We seek to adapt Garcia et al.'s [4] simplest walking model for powered locomotion on level ground, using the methods of McGeer [6] to actuate the stance leg. We then analytically compare the energetics of toe-off vs. hip actuation. This is followed by a study of actuation of the swing leg, which can affect the energy losses that occur at heel strike. Finally, we compare the analytical approximations against numerical calculations using the (fully nonlinear) irreducibly simple model as well as a more anthropomorphic model.

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2 The Powered Walking Model

We extend the simplest walking model of Garcia et al. [4] with the addition of two types of actuation. The original model consists of a point mass representing the pelvis and torso, and two massless legs with a point mass at each foot (see Fig. 1(*a*)). Kinematics are described by stance leg angle θ defined relative to the vertical, and the swing leg angle ϕ defined relative to the stance leg. Taking the limit as the ratio of the masses of each foot, m, to that of the pelvis, M, approaches zero, the model becomes irreducibly simple while still maintaining a passively stable gait [4]. The first type of actuation is that which adds energy to the center of mass and the stance leg. This is accomplished either by extending the leg at toe-off or by applying a torque between the torso and stance leg. We will see that there are energetic advantages associated with a toe-off impulse, applied just before heel strike at the trailing foot and directed at the center of mass. This impulse approximates the extension of the human ankle and knee, triggered by activity of muscles such as the soleus and gactrocnemius, just prior to toe-off [7]. Impulses of this type were analyzed by McGeer [3], and are appropriate for walking on level ground or for climbing slopes.

The second type of actuation is a hip torque between the swing leg and either the torso or stance leg. Observations of humans show a burst of hip extensor activity (e.g., hamstrings) at the end of the swing phase, and a burst of hip flexor activity just after toe-off. We model these bursts with a spring-like hip torque applied to the swing leg. Because the swing foot has negligible mass relative to the pelvis, this torque affects neither the motion of the stance leg nor the overall mechanical energy of the system. We describe this spring by k, the dimensionless torsional spring constant, normalized by body weight and leg length. When $k=0$, the natural frequency is equal to the pendulum frequency of the swing leg alone. Although a spring might be thought to be an inferior model of bursts of hip muscle activity, it is straightforward to show that, for the simplest walking model, any gait produced with a spring can be mapped directly to an equivalent gait produced by burst-like hip torques (see Appendix A). In addition, all relevant

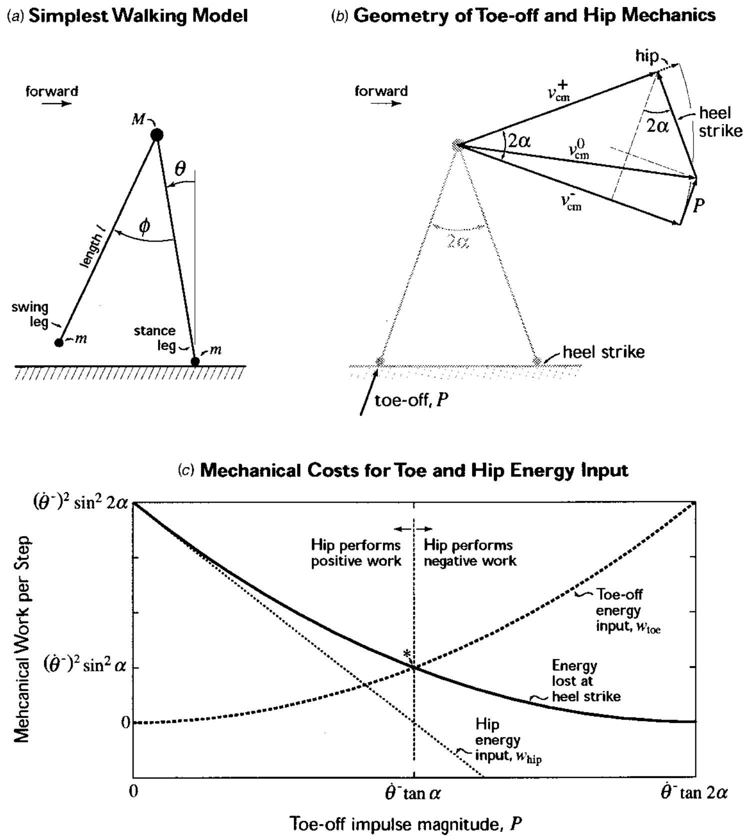

Fig. 1 (*a*) Configuration variables for the simplest walking model of Garcia et al. [4]. (*b*) The velocities before toe-off, after toe-off, and after heel strike (v_{cm}^-, v_{cm}^0, and v_{cm}^+, respectively) can be related by a simple geometric diagram, yielding Eq. (10). At the end of a step, the center of mass has a velocity v_{cm}^- that is perpendicular to the trailing leg. Immediately before heel strike, a toe-off impulse *P* redirects the center of mass velocity v_{cm}^0 along a line parallel to the trailing leg. The heel strike impulse then reduces the velocity to v_{cm}^+, which is perpendicular to the leading leg. The relatively low masses of the feet implies that each impulse can be described by geometric projection along one of the legs. (*c*) Energy relationships needed to sustain the same speed *v* as a function of toe-off impulse magnitude, *P*. Energy must be supplied wholly by a hip torque on the stance leg when $P=0$, but that quantity decreases linearly with increasing *P*. The work done by toe-off increases quadratically with *P* (Eq. 11), and the energy lost at heel strike decreases quadratically. Discounting the possibility of regeneration of negative work by the hip, the least costly gait is achieved by providing all the energy with toe-off (marked with asterisk).

variables associated with burst-like torques, such as peak muscle force and potential energy, are approximately proportional to the corresponding variables from the spring model.

The conservative spring model is employed for conceptual simplicity, but is not meant to imply that actuation of the swing leg comes at no metabolic cost. The possible metabolic costs associated with swing leg actuation are explored in a separate paper [8].

The following equations will be expressed in dimensionless terms, using overall mass M, leg length l, and time $t \triangleq \sqrt{l/g}$ as base units. Velocities will therefore be made dimensionless by factor $\sqrt{gl}$, equivalent to the square root of the Froude number [9]. The equations of motion are derived by straightforward extension of Garcia et al. [4], with the addition of the hip spring:

$$\ddot{\theta}(t) - \sin\theta(t) = 0 \tag{1}$$

$$\ddot{\phi}(t) - \ddot{\theta}(t) - \dot{\theta}^2(t)\sin\phi(t) + \cos\theta(t)\sin\phi(t) = -k\phi(t). \tag{2}$$

To a second-order approximation, these equations may be linearized about $\theta=0$, $\dot{\theta}=0$ to yield

$$\ddot{\theta}(t) - \theta(t) = 0 \tag{3}$$

$$\ddot{\phi}(t) - \ddot{\theta}(t) + \omega^2\phi(t) = 0, \tag{4}$$

where ω is defined by

$$\omega \triangleq \sqrt{k+1}. \tag{5}$$

A further simplification discovered by Garcia et al. [4] is that with masses concentrated at the hip and feet, the boundary conditions for the swing leg can be determined from the stance leg alone. With initial values $\theta(0)=\alpha$ and $\dot{\theta}(0)=\Omega$, either set of differential equations can be integrated forward in time, ending at heel strike when the swing leg comes in contact with the ground when

$$\phi(\tau)-2\theta(\tau)=0, \tag{6}$$

where τ is the step period. Following McGeer [3], we neglect scuffing of the foot during mid-swing, as that can be avoided with an infinitesimal lifting of the foot.

The collision with ground is modeled as instantaneous and perfectly inelastic. McGeer [3] showed that the state following impact can be found using conservation of angular momentum and impulse–momentum equations, which Garcia et al. [4] applied to the simplest walking model. Their equations are easily modified to accommodate the toe-off impulse P applied to the stance foot and directed at the center of mass (see Fig. 1(*a*)). We apply the impulse immediately before heel strike; it can also be applied afterwards, but the latter choice is more costly [6]. The jump conditions are

$$\begin{bmatrix}\theta\\ \dot{\theta}\\ \phi\\ \dot{\phi}\end{bmatrix}^{+}=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & \cos 2\theta & 0 & 0\\ -2 & 0 & 0 & 0\\ 0 & \cos 2\theta(1-\cos 2\theta) & 0 & 0\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\\ \phi\\ \dot{\phi}\end{bmatrix}^{-}+\begin{bmatrix}0\\ \sin 2\theta\\ 0\\ (1-\cos 2\theta)\sin 2\theta\end{bmatrix}P, \tag{7}$$

where we have also switched the stance and swing legs for the beginning of the next step, and the superscripts "$-$" and "$+$" refer to the state immediately before impulse and after impact, respectively. The angles θ and ϕ refer to configuration at time τ using the naming convention prior to impulse. The impulse P is dimensionless with normalization factor $M\sqrt{gl}$.

In order for the walking cycle to be periodic, it is necessary for the composition of the preceding equations to yield a new set of initial conditions for a following step that are equal to the original conditions. The boundary conditions are

$$\theta^{+}(\tau)=\theta(0)=\alpha \tag{8}$$

$$\dot{\theta}^{+}(\tau)=\dot{\theta}(0)=\Omega. \tag{9}$$

3 Mechanical Work on the Center of Mass and Stance Leg

To achieve a steady gait on level ground it is necessary to add sufficient mechanical energy to the center of mass to overcome energetic losses incurred at heel strike. We are considering two methods: an impulse applied to the stance foot immediately before heel strike, and a hip torque applied to the stance leg during single support. McGeer [6] studied both cases; however, when applied to the simplest walking model the results are particularly insightful.

The toe-off impulse P causes an instantaneous increase in the vertical component of center of mass velocity, and the subsequent heel strike causes an instantaneous decrease. The linear momentum-impulse equation, with v_{cm} referring to the magnitude of the center of mass velocity (made dimensionless by $\sqrt{gl}$), is

$$v_{\mathrm{cm}}^{+}=v_{\mathrm{cm}}^{-}\cos 2\alpha+P\sin 2\alpha. \tag{10}$$

This equation is equivalent to the second row of Eq. (7). The mechanical work per step done by this toe-off impulse is

$$w_{\mathrm{toe}}=\tfrac{1}{2}P^2. \tag{11}$$

A periodic cycle can be achieved either by using a toe-off impulse of sufficient magnitude, or by using any combination of toe-off impulse and a hip torque applied to the stance leg from the torso (as studied in [6]). The primary requirement is that together, the toe-off and hip actuators provide sufficient energy to sustain steady gait. Without need to specify the exact time course of the hip torque, a simple energy balance shows that the net work that must be performed by the hip torque must therefore be

$$\begin{aligned} w_{\mathrm{hip}}&=\tfrac{1}{2}(v_{\mathrm{cm}}^{-})^2-\tfrac{1}{2}(v_{\mathrm{cm}}^{+})^2\\ &=\tfrac{1}{2}(v_{\mathrm{cm}}^{-})^2-\tfrac{1}{2}(v_{\mathrm{cm}}^{-}\cos 2\alpha+P\sin 2\alpha)^2. \end{aligned} \tag{12}$$

The total work performed is

$$w_{\mathrm{toe}}+w_{\mathrm{hip}}=\tfrac{1}{2}[P^2\cos^2 2\alpha-P(v_{\mathrm{cm}}^{-}\sin 4\alpha)+(v_{\mathrm{cm}}^{-})^2\sin^2 2\alpha]. \tag{13}$$

This equation offers an analytical insight to powered walking. Eq. (13) is quadratic in P (see Fig. 1(*c*)), and assuming that power cannot be regenerated from heel strike, the total work is minimized with $P=v_{\mathrm{cm}}^{-}\tan\alpha$, when the toe-off impulse is supplying all of the energy. Comparing two extremes, using the hip alone is 4 times more expensive than using toe-off alone. This is because hip powering increases the collision loss at heel strike, while toe-off powering decreases this loss. Using small angle approximations and $v\approx v_{\mathrm{cm}}^{-}$, the mechanical energy cost per step is about $w_{\mathrm{hip}}=2v^2\alpha^2$ for hip actuation and $w_{\mathrm{toe}}=\frac{1}{2}v^2\alpha^2$ for toe-off actuation. The cost per distance (denoted by upper case "W") is $W_{\mathrm{hip}}=v^2\alpha$ for hip actuation and $W_{\mathrm{toe}}=\frac{1}{4}v^2\alpha$ for toe-off actuation.

Therefore, we find that the optimal strategy for powering the simple walker is to apply toe-off actuation alone. The impulse must be applied immediately before heel strike in order to reduce the collision—an impulse applied earlier in the step [6] suffers the same disadvantage as hip actuation. Even if both toe-off and hip actuation are used, as long as the energy supplied by each method is kept in a fixed proportion, the energy consumed per unit distance is proportional to $v^2\alpha$. It is also equal to the energy dissipated at heel strike (as long as the hip is not allowed to absorb energy), and energetic cost per distance is lowest when toe-off supplies all of the energy.

Whatever the method of actuation, powering the stance leg alone does not guarantee the existence of a limit cycle subject to the constraints of passive walking. Fortunately, McGeer [6] demonstrated both the existence and stability of a limit cycle, as is the case for the simplest walking model. Considering toe-off actuation alone, analysis of limit cycles is aided by the use of the linearized equations of motion (3)–(4), with the jump conditions (7) kept in nonlinear form. Combining these equations with the boundary conditions (6)–(9), we see that α, Ω, τ, ω and P must satisfy (see Appendix B for details) the following equations:

$$\left(\frac{2\omega^2+1}{\omega^2+1}\right)(e^{\tau}-e^{-\tau})(\cos\omega\tau+1)+\frac{1}{(\omega^2+1)\omega}(2+e^{\tau}+e^{-\tau})\sin\omega\tau=0, \tag{14}$$

$$-(e^{\tau}-e^{-\tau})\Omega=(2+e^{\tau}+e^{-\tau})\alpha, \tag{15}$$

$$[2-\cos 2\alpha(e^{\tau}+e^{-\tau})]\Omega=\cos 2\alpha(e^{\tau}-e^{-\tau})\alpha-2P\sin 2\alpha. \tag{16}$$

We will presently consider the case where the swing leg is unactuated, so that $\omega=1$.

Because P does not appear in (14), the step time has theoretically no dependence on the amount of toe-off actuation. Equation (15) also implies that α and Ω will be linearly related for a given step period τ. Assuming small angles, step length (normalized by leg length)

$$s\approx 2\alpha \tag{17}$$

and speed (normalized by $\sqrt{gl}$)

$$v\approx-\Omega \tag{18}$$

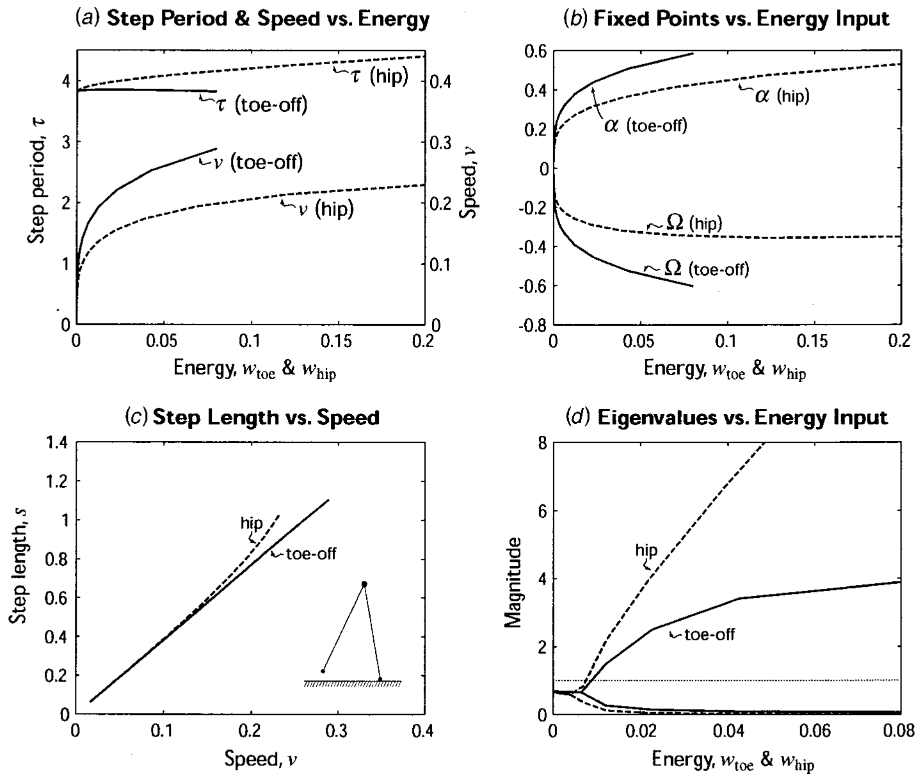

Fig. 2 Simplest walking model gaits produced by toe-off impulses or by hip torques applied to the stance leg from the torso, with the swing leg left unactuated. (*a*) Step period τ (left-hand axis) changes little with input energy for both types of actuation, and especially for toe-off powering. Mechanical work required per step increases with approximately the fourth power of speed *v* (right-hand axis), with hip actuation about four times more expensive than toe-off. (*b*) Periodic gaits, in terms of α and Ω, vary uniformly with energy input. (*c*) The result is that, in the absence of a hip spring to tune the swing leg, the step length *s* increases nearly linearly with speed *v*. (*d*) Eigenvalues of step-to-step transition describe passive stability of gait. For each method of actuation, there are two eigenvalues, of which one is always stable (unmarked curves). Passive stability is lost as energy levels (and speed) increase, although the addition of a hip spring easily brings the eigenvalues to magnitude less than 1.

will also be linearly related. An alternative to solving (16) is to use the approximations (17) and (18), together with the energy calculations, to predict that the work/step done by the toe-off impulse should increase with the fourth power of α or Ω.

Simulations confirm these approximations. We used methods similar to that of McGeer [3] to find limit cycles with the nonlinear equations of motion (2) and (3), using both toe-off and hip actuation. In both cases, the step period τ is practically invariant with respect to energy input, as predicted by (14). The approximation for toe-off powering turns out to be especially accurate, with τ varying about 1 percent over the selected range for P (see Fig. 2(*a*)). The mechanical energy per step is well predicted by $v^2\alpha^2$, with a factor of 4 advantage for toe-off versus hip powering (Figs. 2(*a–b*)), and the step length increases approximately linearly with speed (Fig. 2(*c*)). An interesting discovery is that the simplest walking model is passively unstable for all but the lowest speeds (Fig. 2(*d*)), although we will later see that the addition of a swing leg hip spring restores passive stability. Because of the clear advantages of using toe-off power alone, we will subsequently restrict ourselves to that case. The results that follow are also applicable to hip power, if the factor of 4 increase in the cost of actuating the center of mass is taken into account.

4 Tuning the Swing Leg

Having established that toe-off actuation from the leg is a mechanically advantageous means of providing propulsion, the addition of a spring to actuate the swing leg requires some justification. It is true that there is no sense in adding propulsive energy to the system via the swing leg—indeed, in the simplest walking model the swing leg has negligible mechanical energy relative to the point mass pelvis. Rather, the potential benefit of swing leg actuation is to tune the step frequency and step length, and a spring makes this possible with no net change in the energy per step. In the human, the advantage of adjusting step frequency or length would presumably be offset by the metabolic cost of activating the muscles to behave in a spring-like manner.

A series of simplifications lends considerable insight to the consequences of swing leg actuation. The spring constant will determine the swing leg's natural frequency ω, and from (14), the step period τ. However, rather than solving (14), it is easier to rely on the intuition that the swing leg should always follow a stereotypical pendulum-like trajectory, and that its period τ should be a fairly constant proportion of $1/\omega$,

$$\omega \sim \frac{1}{\tau}. \tag{19}$$

Another complication can be avoided by considering the energetics of toe-off actuation with $P = v_{cm}^{-}\tan\alpha$. When (18) and (19) are used with the assumptions of small angle α, $v \sim v_{cm}^{-}$, and $\Omega \approx 2\alpha/\tau$, it follows that

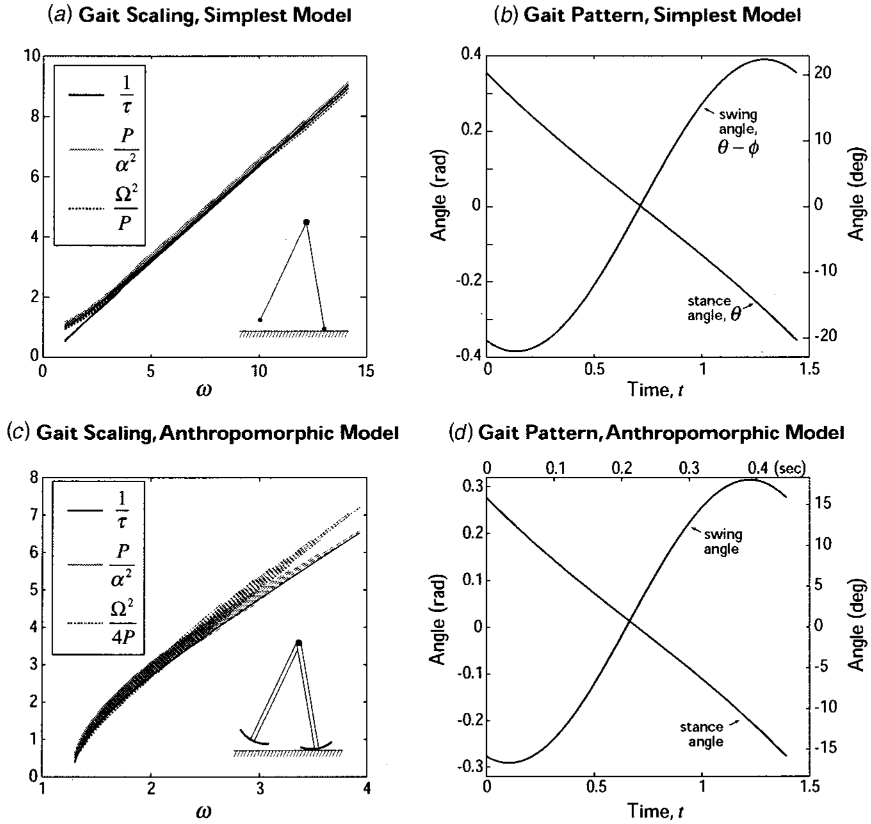

Fig. 3 Scaling relationships for gaits produced by powered passive walking models. The Simplest Model (*a*) approximately obeys predicted relationships (i.e., the lines shown are nearly straight) between swing leg natural frequency ω, toe-off impulse magnitude *P*, and gait variables such as boundary conditions α and Ω and step period τ. The lines show relationships for the entire range of step length and speed combinations simulated (see Fig. 4). Gaits produced (*b*) have two-fold symmetry—the model looks the same walking backward or forward in space and time. Stance and swing angles are reported relative to vertical. When applied to the Anthropomorphic Model with more realistic mass parameters, the scaling relationships (*c*) are followed less well at low ω, and the gaits (*d*) are slightly less symmetrical.

$$\omega \sim \frac{v^2}{P} \tag{20}$$

and

$$\omega \sim \frac{P}{\alpha^2}. \tag{21}$$

These approximations, which we will term the *Idealized Simple Model* (ISM) of powered passive walking, are adequate for qualitative analysis and provide considerable insight. We tested the accuracy of (19)–(21) by performing simulations with the full nonlinear equations for the model of Eqs. (1)–(9)—termed the *Simplest Model* (SM)—and another model similar to McGeer's [3] with roughly anthropomorphic inertial parameters—termed the *Anthropomorphic Model* (AM, described in Appendix C). Over a wide range of values for ω and *P*, both SM and AM obey the scaling predictions of (19)–(21) to a surprising degree (see Fig. 3), albeit with some deviation at small values of ω.

We also calculated the mechanical energy costs required for toe-off actuation for the same range of ω and *P*, using both SM and AM and assuming that the hip spring for the swing leg is conservative (see Fig. 4). With no other energy sinks or sources, toe-off energy is equal to the negative work done at heel strike. The results show that for both models, a wide range of step lengths and speeds can be achieved by appropriate choice of ω and *P*. Exceptions occurred at very low step frequencies or step lengths, where it was difficult or impossible to find limit cycles.

The effect of increasing spring stiffness is to improve walking energetics. As demonstrated by (19)–(21), for a given impulse *P*, step frequency is proportional to ω and speed is proportional to $\sqrt{\omega}$. By decreasing the step length, the spring decreases the collision loss and facilitates a faster speed for the same amount of impulse. Results were similar for both SM and AM, except for two differences. At very high spring stiffnesses the AM tends to increase speed without further decreasing step length. Only the SM was unstable at low step lengths; both models were unstable for low spring stiffnesses.

5 Discussion

There are several implications that can be drawn from the powered walking model. First is the advantage of performing a toe-off push along the stance leg immediately before heel strike. This is a factor of 4 times less costly than alternative means of driving the stance leg either at other times in the gait cycle or via torque at the hip (see Fig. 2). This may place at a disadvantage individuals such as amputees who are limited in their ability to push off with the stance leg. Second, there is an advantage to applying hip torques on the swing leg, not to add propulsive energy but instead to produce forced oscillations in a spring-like manner. This spring-

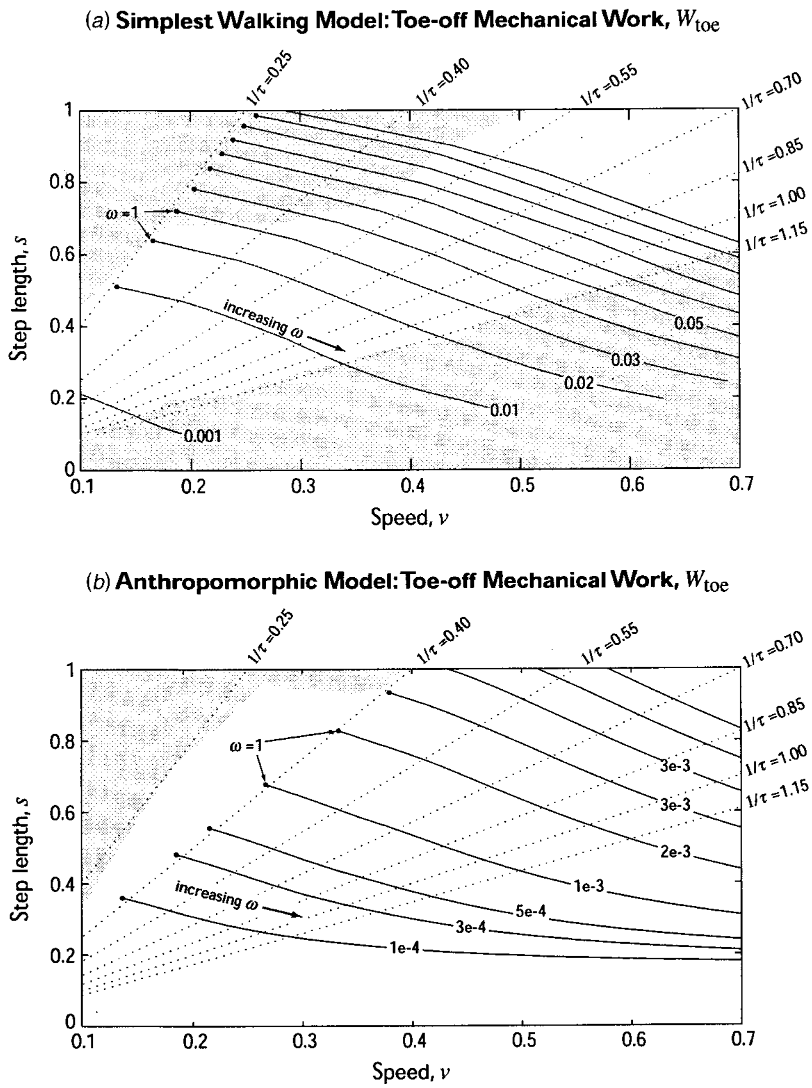

Fig. 4 Mechanical work performed by toe-off, W_{toe}, as a function of speed v and step length, s, using (*a*) the Simplest Model and (*b*) the Anthropomorphic Model. Contour lines of constant (dimensionless) mechanical work per distance (solid lines, with energy levels as labeled) increase with both speed and step length. Gaits achieved with no hip spring acting on the swing leg (ω=1, denoted by filled circles) show that increasing toe-off impulses tend to increase speed and step length in linear proportion. The effect of increasing the natural frequency ω of the swing leg is to increase speed at slightly shorter step lengths. Walking becomes less costly because the hip spring decreases collision losses. Step frequency ($1/\tau$, denoted by dotted lines) is proportional to ω. The Simplest Model has regions that are passively unstable (shaded areas). The Anthropomorphic Model (*b*), using more realistic mass properties, produces results that are very similar, except that it is more difficult to achieve very short steps, and the unstable regions are less prominent. Another feature of AM is that the curved feet lower the collision losses for all gaits.

like actuation lowers the energetic cost by increasing step frequency and by shortening step length (see Fig. 4), which reduces mechanical energy losses incurred at heel strike.

Although not exactly representative of human walking, the impulsive nature of our models of toe-off and heel strike is also instructive for pointing out a potential flaw in conventional measures of external work. Most studies calculate external work on the center of mass based on the combined forces acting on the feet (e.g., [10–13]). The energetic advantages of a toe-off impulse are likely to be realized in the human only if toe-off occurs slightly ahead of, and potentially overlapping with, heel strike. Therefore, it is quite possible that one foot performs positive external work and the other negative work during double support [14]. Because negative work is not likely to be regenerated, conventional measures are likely to underestimate the positive work performed.

These insights arise from the extreme simplicity of the Idealized Simple Model (ISM), which relies on intuition and several linearizing assumptions. The resulting power laws (19)–(20) are useful for deriving general principles of powered walking. Fortunately, these principles are also applicable to the fully nonlinear equations of motion as demonstrated by the Simplest Model, and to the more realistic inertial properties of the Anthropomorphic Model (AM; see Fig. 3).

Although these principles are powerful, it is important that they be tested against more realistic models. Just as the Simplest Model and Anthropomorphic Models verified some aspects of the simplifying assumptions, more complex models are needed to test their applicability and to make more quantitative predictions of actual human walking.

There are several ways that the present models could be made more realistic. McGeer [3] demonstrated that the addition of knees is quite straightforward, retaining passive stability but losing some additional energy due to locking of the knee near the end of the swing phase. This loss was found to be relatively minor but may become quite significant at higher step frequencies. In addition, the inclusion of a plantarflexing ankle, forward-facing feet, and even realistic muscles could all potentially contribute to improved energetic predictions. A more realistic model of the ankle would be useful for exploring the cost of work for the stance leg, and would also help to produce a double-support phase of longer duration. The addition of a massive torso was studied by McGeer [6], and should have little effect on passive dynamic properties other than possibly lowering the mechanical cost per distance due to the advantage of locating the center of mass as high as possible. A further improvement would be to include lateral motion, which imposes additional costs that appear to be fixed with respect to the present models [15]. Of course, these additional features come at some cost in complexity compared to the irreducibly simple model [2]. Simplicity is helpful for exposing fundamental principles, while complexity aids more detailed quantitative study.

Another limitation is that the present models are confined to issues of mechanics and do not address other biological issues. For example, the mechanical benefit of pushing off from the stance leg rather than applying a hip torque may be offset by biological limits on the use of human ankle plantarflexors due to susceptibility to fatigue, constraints on maximum power, and the relationship between mechanical and metabolic costs. Simple models are nevertheless useful for determining whether such biological constraints are active. Mechanics and biology must ultimately be combined to elucidate the mechanisms of gait.

Acknowledgments

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Nomenclature

Dimensional Variables

- M = point mass representing pelvis/torso
- m = point mass representing foot mass
- g = gravitational constant
- l = leg length

Dimensionless Variables

- α = initial condition $\theta(0)$
- ϕ = angle of swing leg with respect to stance leg
- θ = angle of stance leg with respect to vertical
- τ = step period
- ω = natural frequency of swing leg
- Ω = initial condition $\dot{\theta}(0)$
- k = torsional spring constant for swing leg spring
- P = toe-off impulse
- s = step length
- t = dimensionless time
- v = forward walking speed
- v_{cm}^{-}, v_{cm}^{0}, v_{cm}^{+}, = translational speed of center of mass before toe-off, after toe-off but before heel strike, and after heel strike, respectively
- w_{toe} = mechanical work per step performed by toe-off actuation of the stance leg
- w_{hip} = mechanical work per step performed by a hip torque on the stance leg
- W_{toe} = mechanical work per distance performed by toe-off actuation of the stance leg
- W_{hip} = mechanical work per distance performed by a hip torque on the stance leg

Appendix A

Equivalence Between the Hip Spring and Burst-Like Hip Torques. There is a direct correspondence between our model of swing leg motion with a hip spring, and a model with burst-like torques applied on the swing leg. Impulsive torques may be a more physically appealing model of the hip muscle bursts seen in human walking, but the correspondence between impulses and a spring means that results from the mathematically convenient spring model can be readily applied to the impulsive model. The correspondence will be demonstrated for a simple pendulum, and then for the entire gait cycle of the simplest walking model including heel strike and transition between steps.

For simplicity we will first consider action of the swing leg alone, without the forcing term added by the stance leg. The swing leg equation of motion is simply that of a pendulum,

$$\ddot{\phi}+\phi=T_{\mathrm{hip}}. \tag{A1}$$

We wish to actuate the swing leg in such a way as to increase the frequency but without adding any net energy over a step cycle. This can be accomplished by applying a sinusoidal hip torque produced by linear feedback

$$\text{full-duty hip torque:}\quad T_{\mathrm{hip}}=-k\phi, \tag{A2}$$

providing a natural frequency of $\omega=\sqrt{k+1}$. The term "full-duty" refers to the fact that (A2) applies to the entire swing period.

An alternative means of tuning the frequency is to apply the torque only at the extremes of motion, thereby modeling the burst-like nature of hip muscle activity. Defining a constant α_0 to refer to the angle beyond which the torque is active, the corresponding torque law is burst-like hip torque:

$$T_{\mathrm{hip}}=\begin{cases}0 & \text{if } |\phi|<\alpha_0\\ -\operatorname{sgn}(\phi)\tilde{k}(|\phi|-\alpha_0) & \text{if } |\phi|\geqslant\alpha_0\end{cases}. \tag{A3}$$

In order to produce the same amplitude α and period $\tau=2\pi/\omega$, it is necessary to use a higher gain $\tilde{k}$, which would be equivalent to a stiffer spring. The fraction of the period for which the torque is active is given by

$$\text{duty factor}=\frac{4}{\tilde{\omega}\tau}\cos^{-1}\frac{\alpha_0}{\alpha\tilde{\omega}^2+\alpha_0(\tilde{\omega}^2-1)}, \tag{A4}$$

where $\tilde{\omega}=\sqrt{\tilde{k}+1}$. Another equation arises from the need for compatibility at $\phi=\alpha_0$, which requires

$$\text{duty factor}=1-\frac{4}{\tau}\sin^{-1}\frac{\alpha_0}{\sqrt{\tilde{k}(\alpha-\alpha_0)^2+\alpha^2}}. \tag{A5}$$

The parameters α_0 and $\tilde{k}$ may therefore be found so as to produce a gait equivalent (in step length and speed) to that of produced with a full-duty hip torque.

The power law equivalence between a full-duty hip torque and a burst-like torque also applies to the simplest walking model (3), where the forcing term due to the stance leg is included, along with the discontinuities arising from the jump conditions (7). As an example, we demonstrate the equivalence between the simplest model spring and a purely impulsive hip torque, in which the duty factor approaches zero.

Suppose that an impulsive torque (made dimensionless by $ml\sqrt{gl}$), H, is applied directly after heel strike (denoted by superscript "+") and that the symmetrical impulse $-H$ is applied directly before the next toe-off (denoted by superscript "−"). The swing leg velocity is

$$\dot{\phi}^{++}(0)=\dot{\phi}^{+}(0)+H \tag{A6}$$

and

$$\dot{\phi}^{-}(\tau)=\dot{\phi}^{--}(\tau)-H, \tag{A7}$$

where double superscripts "++" and "−−" refer to the moments directly after and before application of torque. Equivalence with a non-zero hip spring is easily achieved, because the swing leg motion has no effect on the stance leg. It is only necessary to achieve the identical $\phi(\tau)$ using either model. For the Simplest Model, (A6), (A7), and (14)–(16) imply that

$$H=\frac{-3\alpha(1+\cos\tau)}{\sin\tau}+\frac{1}{2}\Omega. \tag{A8}$$

The impulse H can therefore be set to match the period τ of an equivalent hip spring model. A particularly simple approximation to (A8) can be found by making the intuitive argument that the spring exerts a force proportional to $k\alpha$ over a duration proportional to τ:

$$H\sim k\alpha\tau \tag{A9}$$

To test this approximation, we performed a regression between H and $k\alpha\tau$ using the numerical results from the entire set of step lengths and speeds explored with the fully nonlinear simplest model (SM, see Fig. 4(*a*)), and found an R^2 of about 0.999.

Appendix B

Details of the Simplest Walking Model With Actuation The algebraic relationship between the parameters for a successful periodic simplest walking gait, Eqs. (14)–(16), are found from the following intermediate result. The solution to the linearized equations of motion (3)–(4) is

$$\theta(t)=\tfrac{1}{2}(\theta(0)+\dot{\theta}(0))e^{t}+\tfrac{1}{2}(\theta(0)-\dot{\theta}(0))e^{-t} \tag{B1}$$

$$\phi(t)=\frac{1}{\omega^2+1}\theta(t)+\frac{2\omega^2+1}{\omega^2+1}\theta(0)\cos\omega t - \frac{1}{\omega(\omega^2+1)}\dot{\theta}(0)\sin\omega t. \tag{B2}$$

Substituting (8) into (B1) yields Eq. (15). Substituting (9) into the second row of (7), and then combining with (B1) yields Eq. (16). Combining (B2) and (B1), along with (15), yields Eq. (14).

Appendix C

The Anthropomorphic Walking Model. To test a more realistic model operating with inertial properties more similar to that of the human, we applied impulsive toe-off and a hip spring to a model similar to McGeer's [3]. Lengths and masses will be given in base units of leg length l and overall mass M, respectively. The model has curved feet with radius 0.3, and legs with centers of mass located $l_c=0.645$ from the feet. Each leg has mass $m=0.16$ and radius of gyration $r_{\text{gyr}}=\sqrt{1/12}$, leaving the pelvis/torso with mass 0.68. Parameter variations were studied extensively by McGeer [3].

The anthropomorphic model's spring constant k was defined so that the swing leg's natural frequency is of a similar form to that of the simplest model (SM). Rather than using base units of mgl, k is made dimensionless by the apparent inertia of the swing leg about the hip, so that the spring constant in physical units is

$$K=k\,\frac{mgr_{\text{gyr}}^2+mg(l-l_c)^2}{l}. \tag{C1}$$

The natural frequency of the swing leg is therefore defined by

$$\omega\equiv\sqrt{k+\frac{(l-l_c)l}{r_{\text{gyr}}^2+(l-l_c)^2}}. \tag{C2}$$

Note that this choice of non-dimensionalizing factors is only relevant to the absolute numerical values given below, and has no bearing on the overall results.

In order to produce a wide range of reasonable walking solutions, we applied toe-off impulses P with magnitude ranging from 0 to 0.22 (base units of $M\sqrt{gl}$), and spring stiffnesses k ranging from 0 to 400. These ranges were chosen to produce speeds of at least 0.7 and step lengths ranging up to at least 1. The results shown in Figs. 3 and 4 were found by using combinations within the preceding ranges, applied in a mesh of at least 20 steps for both P and k.

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